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QUASI-STEADY STATE SOLUTION OF PERIODICALLY VARYING PHENOMENA

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Abstract—In finding the quasi-steady state solution semi-numerical procedures can be advantageous. One such general method is illustrated by means of a heat flow problem with periodically contacting surfaces. The results are compared with those of an analogue study.

NOMENCLATURE

θ, temperature difference;
θ_0, fixed temperature difference of one end of cylinder;
x, distance along cylinder from this end;
l, length of cylinder;
l_i, equivalent length of cylinder;
t, τ, time;
T_1, T_2, contact and separation times, respectively;
k, α, thermal constants of cylinder and film, respectively;
$g(x), h(x)$, temperature distributions at instants of contact and separation;
$\bar{\theta}(n), \bar{g}(n), \bar{h}(n)$, transforms of $\theta(x)$, $g(x)$, $h(x)$ with λ_n as transform parameter;
$\theta^*(m), g^*(m), h^*(m)$, transforms with γ_m as parameter;
s, number of intervals into which cylinder is divided;
g_r, h_r, $g(rl/s)$, $h(rl/s)$;
$c_{i0}, C_{ij}, d_{i0}, D_{ij}$, constants derived in the text;
C, D, $s \times s$ matrices of which C_{ij}, D_{ij} are elements;
c_0, d_0, $s \times 1$ matrices of which c_{i0}, d_{i0} are elements;
p, q, r, f, A, B, F, functions of the independent variable; coefficients in the Sturm–Liouville theory;
μ_m, ϕ_m, mth eigenvalue, eigenfunction respectively;
N_m, mth norm;
P, Q, constants of integration.

INTRODUCTION

THE TYPE of problem to be considered is one which is time dependent but in which, due to periodic external factors, the solution settles down to a periodic form which is then referred to as the quasi-steady state solution. In mathematical terms the same partial differential equation may represent the physical system for all time but the boundary conditions vary in a periodic manner which may or may not be continuous. Since many cycles may be required before the quasi-steady state is reached, numerical techniques alone may prove excessively long and may, in consequence, be inaccurate. If the engineering problem can be idealized in such a way that the only feature preventing an analytical solution is the periodic nature of the boundary conditions, then the semi-numerical procedure to be outlined may be applied.

To present the method as clearly as possible a one-dimensional heat flow problem is considered.

The idealized problem [1], which is not new, is simple to define but not simple to solve; the only solution at present available demands the use of a quite sophisticated analogue technique. Examples of the corresponding engineering problem are the heat transfer from the exhaust valve of an internal combustion engine to its seating, and between a work piece and die in repetitive hot metal deformation processes.

THE HEAT FLOW PROBLEM

In order to illustrate the method as simply as possible the heat flow is restricted to one dimension. This is achieved by considering a solid cylinder of length l with a uniform cross-section whose boundary is insulated so that no heat is lost from the sides of the cylinder. One end is held at a constant temperature θ_0 above a body with which the other end of the cylinder makes periodic contact. This contact is supposed to be made through a film which presents a thermal impedance.

Thus along the length x of the cylinder the temperature θ satisfies

$$\frac{\partial^2\theta}{\partial x^2} = k\frac{\partial\theta}{\partial t} \tag{1}$$

and is subject to the boundary conditions:

(i) at $x = 0$, $\theta = \theta_0$

(ii) if the cylinder is periodically in contact for a time T_1 and separated for a time T_2, then at $x = l$

$$\frac{\partial\theta}{\partial x} = \begin{cases} -\alpha\theta, & 0 < t < T_1 \\ 0, & T_1 < t < T_1 + T_2 \end{cases}$$

where k, α are thermal constants of the cylinder and film respectively.

It will be appreciated that this problem includes that of two identical cylinders periodically contacting through a thermal impedance and whose remote ends differ in temperature by $2\theta_0$.

Time is measured from the start of a cycle after the temperature has achieved its quasi-steady state. It is desirable to introduce two unknown functions $g(x)$, $h(x)$ which are respectively the temperature distributions at the instant of closure $[t = n(T_1 + T_2),\ n = 0, 1, 2 \ldots]$ and separation $[t = T_1 + n(T_1 + T_2)]$. The objective now is to find $g(x)$, $h(x)$. Once these are known the general solution can be found.

The differential equation (1) can be solved by the method of separation of the variables. This method involves a detailed knowledge of Fourier series and the amount of algebraic work involved can be greatly reduced by using the knowledge in the form of a finite integral transform [3]. Since moreover a reduction in manipulation leads to greater freedom from errors the latter method is adopted here.

It is only necessary to consider the two intervals of one cycle, namely

$$0 < t < T_1, \qquad T_1 < t < T_1 + T_2.$$

In the first interval the boundary conditions are:

$$\text{at } x = 0, \qquad \theta = \theta_0$$

$$\text{at } x = l, \qquad \partial\theta/\partial x = -\alpha\theta$$

for which a suitable finite transform (in which θ is replaced by $\bar{\theta}$) is developed in the Appendix as

$$\bar{\theta}(n) = \int_0^l \theta(x)\sin(\lambda_n x)\,dx; \qquad \tan(\lambda_n l) = -\lambda_n/\alpha$$

with inverse

$$\theta(x) = \sum_{n=1}^{\infty}\frac{\bar{\theta}(n)\sin(\lambda_n x)}{N_n}; \qquad N_n = \frac{\alpha l + \cos^2(\lambda_n l)}{2\alpha}.$$

On multiplying equation (1) by $\sin(\lambda_n x)$ and integrating over $(0, l)$ there results

$$\frac{d\bar{\theta}}{dt} + \frac{\lambda_n^2\bar{\theta}}{k} = \frac{\lambda_n\theta_0}{k} \tag{2}$$

the solution of which is

$$\bar{\theta}\exp(\lambda_n^2 t/k) - \bar{g}(n) = \theta_0\{\exp(\lambda_n^2 t/k) - 1\}/\lambda_n \tag{3}$$

with

$$\bar{g}(n) = \int_0^l g(x)\sin(\lambda_n x)\,dx \tag{4}$$

as the corresponding transform of the (as yet) unknown function $g(x)$.

In particular at $t = T_1$, $\bar{\theta} = \bar{h}(n)$ so that

$$\bar{h}(n)\exp(\lambda_n^2 T_1/k) - \bar{g}(n) = \theta_0\{\exp(\lambda_n^2 T_1/k) - 1\}/\lambda_n \quad (5)$$

the inverse of $\bar{h}(n)$ being

$$h(x) = \theta_0 \sum_{n=1}^{\infty} \frac{\{1 - \exp(-\lambda_n^2 T_1/k)\}\sin(\lambda_n x)}{\lambda_n N_n} + \sum_{n=1}^{\infty} \frac{\{\bar{g}(n)\exp(-\lambda_n^2 T_1/k)\}\sin(\lambda_n x)}{N_n}. \quad (6)$$

Divide the range $(0, l)$ into s equal intervals and let

$$g(rl/s) = g_r, \qquad h(rl/s) = h_r, \qquad 0 \leqslant r \leqslant s$$

then, using the trapezoidal rule to approximate to $\bar{g}(n)$;

$$\bar{g}(n) = \sum_{j=1}^{s} K_j g_j \sin(\lambda_n jl/s) \quad (7)$$

where

$$K_j = \begin{cases} l/s, & j \neq s \\ l/(2s), & j = s. \end{cases}$$

Equation (6) may now be written for $1 \leqslant i \leqslant s$

$$h_i = c_{i0} + \sum_{j=1}^{s} C_{ij} g_j \quad (8)$$

in which

$$c_{i0} = \theta_0 \sum_{n=1}^{\infty} \frac{\{1 - \exp(-\lambda_n^2 T_1/k)\}\sin(\lambda_n il/s)}{\lambda_n N_n}$$

and for $1 \leqslant j \leqslant s$

$$C_{ij} = \sum_{n=1}^{\infty} \frac{K_j}{N_n}\{\exp(-\lambda_n^2 T_1/k)\} \times \sin(\lambda_n il/s)\sin(\lambda_n jl/s),$$

all these c, C's being known. In matrix form equation (8) may be written

$$h = c_0 + Cg. \quad (9)$$

In the second time interval $T_1 < t < T_1 + T_2$ it is convenient to let

$$\tau = t - T_1$$

so that, in the interval $0 < \tau < T_2$

$$\frac{\partial^2\theta}{\partial x^2} = k\frac{\partial\theta}{\partial\tau}$$

with,
at $x = 0$, $\theta = \theta_0$;
at $x = l$, $\partial\theta/\partial x = 0$;
at $\tau = 0$, $\theta = h(x)$;
at $\tau = T_2$, $\theta = g(x)$.

The appropriate finite transform (see Appendix) is

$$\theta^*(m) = \int_0^l \theta(x)\sin(\gamma_m x)\,dx, \qquad \gamma_m = (2m+1)\pi/(2l)$$

with inverse

$$\theta(x) = (2/l)\sum_{m=0}^{\infty} \theta^*(m)\sin(\gamma_m x)$$

which, when applied, yields

$$\frac{d\theta^*}{d\tau} + \frac{\gamma_m^2\theta^*}{k} = \frac{\gamma_m\theta_0}{k}$$

the solution to which is

$$\theta^*(m)\exp(\gamma_m^2\tau/k) - h^*(m) = \theta_0\{\exp(\gamma_m^2\tau/k) - 1\}/\gamma_m. \quad (10)$$

In particular when $\tau = T_2$, $\theta^*(m) = g^*(m)$, so that

$$g^*(m) = \theta_0\{1 - \exp(-\gamma_m^2 T_2/k)\}/\gamma_m + h^*(m)\exp(-\gamma_m^2 T_2/k)$$

the inverse of which is

$$Kg(x) = \frac{2\theta_0}{l}\sum_{m=0}^{\infty}\frac{\{1 - \exp(-\gamma_m^2 T_2/k)\}\sin(\gamma_m x)}{\gamma_m} + \frac{2}{l}\sum_{m=0}^{\infty} h^*(m)\exp(-\gamma_m^2 T_2/k)\sin(\gamma_m x). \quad (11)$$

Again the trapezoidal rule is used to approximate to $h^*(m)$ in the form

$$h^*(m) = \sum_{j=1}^{s} K_j h_j \sin(\gamma_m jl/s) \tag{12}$$

so that equation (11) may be written

$$g_i = d_{i0} + \sum_{j=1}^{s} D_{ij} h_j$$

where

$$d_{i0} = \frac{2\theta_0}{l} \sum_{m=0}^{\infty} \frac{1 - \exp(-\gamma_m^2 T_2/k)}{\gamma_m} \sin(\gamma_m il/s)$$

and for $1 \leqslant j \leqslant s$

$$D_{ij} = (2/l) \sum_{m=0}^{\infty} K_j \exp(-\gamma_m^2 T_2/k) \times \sin(\gamma_m il/s) \sin(\gamma_m jl/s).$$

In matrix form

$$g = d_0 + Dh \tag{13}$$

which, when used in conjunction with equation (9) gives

$$g = [I - DC]^{-1} [Dc_0 + d_0]$$
$$h = [I - CD]^{-1} [Cd_0 + c_0]$$

where I is the unit matrix.

The temperature distributions at the instants of closure and separation, $g(x)$ and $h(x)$ respectively are now known and since these provide the bounds between which the temperature varies, it is generally sufficient. However if the temperature at other times is required it can be obtained by finding $\bar{g}(n)$, $h^*(m)$ from equations (7), (12) and then inverting equations (3), (10).

EXAMPLE

Sutton and Howard have extended their analogue work [1] to include a contact film so that it was natural to choose an example for which their results were available (although as yet unpublished [2]). Such an example is provided by the following data

$l = 4 \times 10^{-2}$ m $\quad T_1 = T_2 = 25$ s

$k = 0{\cdot}2 \times 10^6$ sm^{-2} $\quad \alpha = 5 \times 10^2$ m^{-1}

$\theta_0 = 1$.

Using $s = 10$ intervals the resulting temperature distributions $g(x)$, $h(x)$ are shown in Fig. 1.

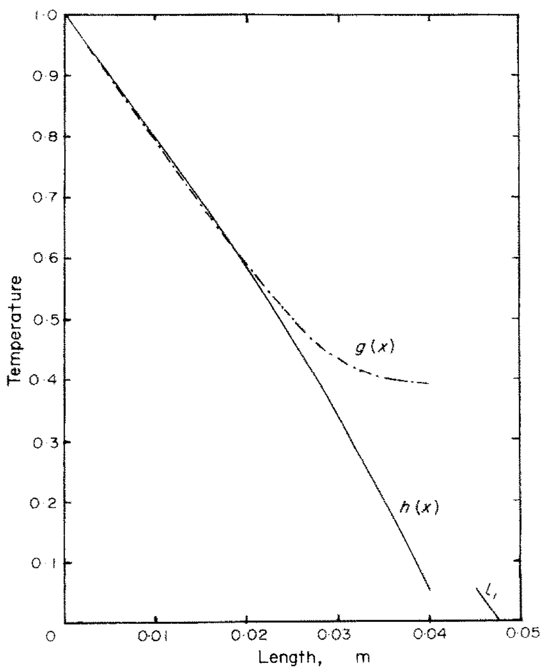

FIG. 1.

An equivalent length l_i of cylinder may be defined as that in which, when its ends are held at temperature θ_0 and 0 for all time, the average heat flow from the source is the same as in the original cylinder. This equivalent length is then obtained in terms of the gradient of the time average, $-M$, (which can be obtained at the fixed temperature end) as

$$l_i = \theta_0/M.$$

In the example M is estimated from Fig. 1 as 21 so that l_i is 0·0476 m which is within 1 per cent of that obtained by the analogue method.

If in some cases the $g(x)$, $h(x)$ distributions are separated even at the fixed temperature end then to obtain an accurate estimate it is necessary to calculate the time-average distribution using the equations already derived. Usually however sufficient accuracy can be achieved by graphical interpolation.

Since the method involves the inversion of an $s \times s$ matrix, where s is the number of sub-divisions, limits will be placed on the value of s by the computer available. This problem occurs only when the cylinder is long in the sense that g, h are identical over a large part (say >70 per cent) of the cylinder. If the temperature half-way along the cylinder is found to be θ_1 say, then the original problem is replaced with that of a cylinder half the length with the end held at θ_1 and is then sub-divided into s parts. For very long cylinders the process can be repeated.

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APPENDIX

General Theory

The construction of finite transforms is illustrated using a secord order ordinary differential equation which, without loss of generality, can be expressed in the Sturm–Liouville form

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y = f(x) \tag{14}$$

and this inhomogeneous equation is to be solved in the interval $a \leqslant x \leqslant b$ subject to the inhomogeneous boundary conditions

$$A(x)\frac{dy}{dx} + B(x)y = F(x), \qquad \text{at } x = a, x = b \tag{15}$$

in which A and B are not zero together.

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A homogeneous form of equation (14) is

$$\frac{d}{dx}\left\{p\frac{d\phi}{dx}\right\} + q\phi = \mu r(x)\phi \tag{16}$$

which is solved subject to the corresponding homogeneous boundary conditions

$$A\frac{d\phi}{dx} + B\phi = 0, \qquad \text{at } x = a, x = b. \tag{17}$$

Then, in the non-degenerate case, non-trivial solutions ot equation (16) which satisfy equation (17) exist only when μ takes one of a set $\{\mu_n\}$ of values (the eigenvalues) and to each μ_m there corresponds a unique ϕ_m (the eigenfunctions). This means that

$$\frac{d}{dx}\left(p\frac{d\phi_m}{dx}\right) + q\phi_m = \mu_m r\phi_m \tag{18}$$

$$\frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n = \mu_n r\phi_n \tag{19}$$

so that, on multiplying equations (18) and (19) by ϕ_n, ϕ_m respectively, subtracting and integrating over (a, b)

$$(\mu_m - \mu_n)\int_a^b r\phi_n\phi_m \, dx = \left[p\left(\phi_n\frac{d\phi_m}{dx} - \phi_m\frac{d\phi_n}{dx}\right)\right]_a^b$$

$$= 0, \qquad m \neq n$$

on using equation (17); when $n = m$ we set

$$\int_a^b r\phi_m^2 \, dx = N_m.$$

It follows that any function $g(x)$ satisfying Dirichlet's conditions in the interval $a \leqslant x \leqslant b$, can be expanded in the form

$$\left.\begin{aligned} g(x) &= \sum_{m=1}^{\infty} \bar{g}(m)\phi_m/N_m \\ \text{where} \qquad \bar{g}(m) &= \int_a^b r\phi_m g \, dx \end{aligned}\right\} \tag{20}$$

and this pair of equations defines a finite transform and its inverse.

To solve equation (14) multiply it throughout by ϕ_m and subtract equation (18) multiplied by y to give, on integration

$$\mu_m\bar{y}(m) = \int_a^b f(x)\phi_m(x)\, dx - \left[p\left(\phi_m\frac{dy}{dx} - y\frac{d\phi_m}{dx}\right)\right]_a^b$$

from which $\bar{y}(m)$ can be found and hence $y(x)$ on using the inverse given in equation (20).

Application to Present Case

In this paper a transform is taken with respect to x and the homogeneous equation corresponding to equation (1) is

$$\frac{\partial^2 \phi}{\partial x^2} = -\lambda^2 \phi \qquad (21)$$

taking $r(x) \equiv 1, \mu = -\lambda^2; a = 0, b = l$. The general solution of equation (21) is

$$\phi = P \sin \lambda x + Q \cos \lambda x, \qquad P, Q \text{ constants.}$$

The corresponding homogeneous form of the boundary conditions for θ are included in those specified for ϕ in equation (17).

In the Interval $0 < t < T_1$

At $x = 0$, $\quad \phi = 0$ so that $Q = 0$;

$$\text{at } x = l, \qquad \frac{\partial \phi}{\partial x} + \alpha\phi = 0$$

so that

$$\lambda \cos \lambda l + \alpha \sin \lambda l = 0, \qquad \text{i.e. } \tan \lambda l = -\lambda/\alpha.$$

The (positive) roots of this last equation are the eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

From the general theory

$$\int_0^l \sin(\lambda_n x) \sin(\lambda_m x)\, dx = 0, \qquad m \neq n;$$

if $m = n$ it is easy to show that

$$\int_0^l \sin^2(\lambda_m x)\, dx = \frac{\alpha l + \cos^2(\lambda_m l)}{2\alpha} \equiv N_m. \qquad (22)$$

Hence the transform pair is given by equation (20) with

$$\phi_m = \sin(\lambda_m x) \quad \text{and} \quad N_m$$

given by equation (22).

In the interval $T_1 < t < T_1 + T_2$

At $x = 0$, $\quad \phi = 0$ so that $Q = 0$;

at $x = l$, $\quad \partial\phi/\partial x = 0$ so that $\cos(\lambda l) = 0$

i.e.

$$\lambda_m = (2m + 1)\pi/(2l)$$

are the eginvalues, the corresponding eigenfunctions are $\sin\{(2m + 1)\pi x/(2l)\}$, $N_m = l/2$ and hence the transform pair can be deduced from equation (20).

SOLUTION QUASI-STATIONNAIRE DE PHENOMENES PERIODIQUES

Résumé—Dans la recherche d'une solution quasi-stationnaire, des procédés semi-numériques peuvent être avantageux. Une méthode générale est illustrée par un problème d'écoulement de chaleur pour des surfaces en contact périodique. Les résultats sont comparés à ceux d'une autre étude.

QUASISTATIONÄRE LÖSUNG FÜR PERIODISCHE VERÄNDERLICHE PHÄNOMENE

Zusammenfassung—Bei der Lösung eines quasistationären Zustandes können halbnumerische Verfahren vorteilhaft sein. Solch eine allgemeine Methode wird an Hand eines Wärmeleitungsproblems mit periodisch sich berührenden Oberflächen gezeigt. Die Ergebnisse werden mit einer Analogiestudie verglichen.

КВАЗИСТАЦИОНАРНОЕ РЕШЕНИЕ ПЕРИОДИУЕСКИ ИЗМЕНЯЮЩИХСЯ ПРОЦЕССОВ

Аннотация—Для нахождения квазистационарного решения могут успошно применяться численные методы. Один из таких методов иллюстрируется на примере решения задауи о теплообмене периодически контактирующих поверхностей. Результаты сравниваются с даннвми, полученными аналоговым методом.